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A NOTE ON RATIONAL THREATS AND COMPETITIVE EQUILIBRIUM

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ABSTRACT

Many economic problems can be modeled as n-person nonzero-sum games. In such situations there are gains to be had by coordination of strategies. Assuming there are no restrictions on side payments, the players then bargain over the division of the gains. This note establishes that, for a restricted class of economic problems, the threat equilibrium in the bargaining game coincides with the perfectly competitive equilibrium.

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I. Introduction

In economic analysis we are often concerned with problems which can be modeled as n-person, nonzero-sum (NPNZS) games. For example, there are oligopoly games, games of research and development, and games of resource extraction (caused by common property externalities). In such situations it is possible to generate greater joint profits by coordinating the choice of strategies rather than playing noncooperatively. If there are no restrictions on side payments, we would expect to see such strategy coordination.

Consider a situation in which there are gains to cooperation and the parties agree to maximize joint profits and then to submit to arbitration to divide the profits. They know that the arbitrator will split the profits using the Nash Bargaining Solution (NBS) [Nash, 1953]. In the NBS, the division of profits depends upon some reference point. That is, for the game with side payments, the NBS is

$$\varphi_i = \frac{V^\circ + (n-1)V^i - \sum_{m \neq i} V^m}{n}, \qquad i = 1,2,\ldots,n$$

where $V^\circ \equiv$ maximized joint profits and $(V^1, V^2, \ldots, V^n)$ is the disagreement or reference point, an allocation generated by noncooperative play.

Nash (1953), and Harsanyi (1959) have argued that since the players intend to maximize joint profits and are merely bargaining over the split, it is reasonable that they should behave strategically in determining the reference point. That is, player i would prefer to choose a (threat) strategy s_i so as to maximize $\varphi_i(s_1, \ldots, s_n)$. Thus the original game has generated another "threat game."

I will refer to the Nash equilibrium for the threat game as a "rational threats equilibrium."

Thus the players report their threat strategies (what they will do should negotiations break down) to the arbitrator. The arbitrator then calculates the reference point and implements the NBS. I am interested in the structure of these equilibrium threats.

Since static games are degenerate versions of dynamic games, I will focus on differential games or games played in continuous time.[1]

II. The Proposition

In this note, I wish to focus on n-person games with a particular structure. Let x denote an n-tuple of state variables and s an n-tuple of controls. Suppose the players have identical strategy spaces and symmetric payoffs. To formalize this notion of symmetry, let the players' index set be $I = \{1,2,\ldots,n\}$. Now define $\pi: I \to I$ to be a permutation of the index set I. There are n! such permutations, indexed by $k \in K = \{1,2,\ldots,n!\}$. Thus x_{π_k} denotes the vector of state variables with indices permuted according to the k^{th} permutation π_k.

Define the function $g = g(t,x,s)$ where

$$g(t,x_{\pi_k},s_{\pi_k}) = g(t,x_{\pi_{k'}},s_{\pi_{k'}})$$

for all k and k' in K.

That is, the function g looks the same for all permutations of the index set I.

Similarly, define $f_i = f(t,x,s_i)$ where

$$f(t,x_{\pi_k},s_i) = f(t,x_{\pi_{k'}},s_i)$$

for all k and k' in K.

Finally let $h_i = h(t,s_i,s/i)$ where $s/i = (s_1,\ldots,s_{i-1}, s_{i+1},\ldots,s_n)$. Suppose that

$$h(t,s_i,(s/i)_{\pi_k}) = h(t,s_i,(s/i)_{\pi_{k'}})$$

for all k and $k' \in K$. Assume that g, f_i and h_i are continuous in t and continuously differentiable in (x,s). Then we restrict the payoff structure to be of the following form: $\forall$ i,

$$V^i(s) = \int_0^T [gs_i - f_i]dt \tag{1}$$

where $\dot{x}_i = h_i + \delta(t)x_i$, $x_i(0) = x_0$, and $\delta(t)$ is continuous on [0,T]. The corresponding threat game payoffs are

$$\varphi_i(s) = V^\circ/n + (1/n)\int_0^T [(n-1)\{gs_i - f_i\} - \sum_{m\neq i}\{gs_m - f_m\}]dt.$$

If we define the strategy space for i to be the set of open-loop strategies (or piecewise continuous functions $s_i(t)$ on [0,T]) then we can apply optimal control theory. The Hamiltonian for i is

$$H_i = \frac{(n-1)}{n}[g\,s_i - f_i] - 1/n\sum_{m\neq i}[g\,s_m - f_m] + \sum_{m=1}^{n}\lambda_m^i[h_m + \delta(t)x_m]$$

where λ_m^i are time-dependent Lagrange multipliers and H_i is assumed jointly concave in (x,s_i).

Each player i chooses $s_i(t)$ so as to maximize H_i, taking $s_j(t)$ as given for all $j \neq i$. When an equilibrium n-tuple exists, it satisfies the usual conditions:[2]

$$\frac{\partial H_i}{\partial s_i} = \frac{(n-1)}{n}\left[\frac{\partial g}{\partial s_i}\cdot s_i + g - \frac{\partial f_i}{\partial s_i}\right] - 1/n\sum_{m\neq i}\frac{\partial g}{\partial s_i}\cdot s_m + \sum_{m=1}^{n}\lambda_m^i\frac{\partial h_m}{\partial s_i} = 0 \tag{2}$$

$$\dot{\lambda}_j^i = -\frac{\partial H_i}{\partial x_j} = -\left\{\frac{(n-1)}{n}\left[\frac{\partial g}{\partial x_j}\cdot s_i - \frac{\partial f_i}{\partial x_j}\right] - 1/n \sum_{m\neq i}\left[\frac{\partial g}{\partial x_j}\cdot s_m - \frac{\partial f_m}{\partial x_j}\right]\right.$$

$$\left. + \lambda_j^i \delta(t)\right\}, \quad \lambda_j^i(T) = 0 \tag{3}$$

Since the players are entirely symmetric, it is reasonable to look for a symmetric threat equilibrium. Then $s_i^* = s_j^*$ for all i and j, and (2)-(3) reduce to

$$\frac{(n-1)}{n}\left[g - \frac{\partial f_i}{\partial s_i}\right] + \sum_{m=1}^{n} \lambda_m^i \frac{\partial h_m}{\partial s_i} = 0 \tag{4}$$

$$\dot{\lambda}_j^i = -\lambda_j^i \delta(t), \quad \lambda_j^i(T) = 0 \tag{5}$$

But (5) means that $\lambda_j^i(t) = c_j^i \exp\left\{-\int^t \delta(\tau)d\tau\right\}$. Since $\exp\{\cdot\}$ is never **zero**,[3] $\lambda_j^i(T) = 0$ implies that the constant $c_j^i = 0$ for all i,j.

Thus we are left with the equilibrium condition

$$g = \frac{\partial f_i}{\partial s_i}, \qquad i = 1,2,\ldots,n \tag{6}$$

In a variety of settings, (6) can be interpreted as a first-order condition for an instantaneously competitive equilibrium. That is, when the problem has the structure as summarized in (1) the threat equilibrium coincides with the instantaneously competitive (not merely Nash-Cournot) equilibrium.

III. <u>Examples</u>

Several examples may illustrate the nature of the restrictions, the types of economic problems which satisfy them, and the interpretation of condition (6).

1.

Consider an intertemporal game of oligopoly with durable goods. Then the price obtainable at t for a unit of current output q_i depends upon i's current output, everyone else's current output and the existing stock of the durable good. That is (although we assume that there is no resale market), current demand is reduced the greater is the stock outstanding, since durable goods are purchased only periodically. Thus the current price is $p(t,Q,\Sigma q_j)$ and we assume that the evolution of the stock Q is governed by

$$\dot{Q} = \Sigma q_j - \delta(t)Q$$

where $\delta(t)$ is the rate at which the stock Q depreciates, wears out, or is consumed.

If the production costs are $c(t,q_i)$ the payoffs become

$$V^i(q) \equiv \int_0^T e^{-rt}[p(t,Q,\Sigma q_j)q_i - c(t,q_i)]dt$$

where $\dot{Q} = \Sigma q_j - \delta(t)Q$, $Q(0) = Q_o$.

In light of the foregoing analysis, the threat equilibrium is $(q_1^*,\ldots,q_n^*)$ such that

$$p(t,Q,\Sigma q_j^*) = \frac{\partial c(t,q_i^*)}{\partial q_i}, \qquad i = 1,\ldots,n$$

But these characterize instantaneously competitive behavior (price = marginal cost). So n identical oligopolists, in bargaining over the division of jointly maximized profits, will specify as their threat (or default) strategies the (instantaneously) competitive equilibrium strategies.

2.

Consider the game of research and development discussed in Reinganum (1979). The payoffs are

$$V^i(u_1,\ldots,u_n) = \int_0^T e^{-\lambda\Sigma z_j}[P\lambda u_i - e^{-rt}u_i^2/2]dt$$

where $\dot{z}_i = u_i$, $z_i(0) = z_o$, $i = 1,2,\ldots,n$.

This game was formulated from the following scenario: if no firm has succeeded in completing the innovation by t (which occurs with probability $e^{-\lambda\Sigma z_j}$), then each firm invests at the (discounted) rate \$ $e^{-rt}u_i^2/2$ to increase its knowledge stock by u_i. In addition, if no firm has yet succeeded, there is an instantaneous conditional probability of success of λu_i. Success is rewarded by a patent with value \$P.

The structure of this game is clearly subsumed by the conditions in (1). Thus at the rational threats equilibrium,

$$u_i^* = P\lambda e^{rt}.$$

But these are precisely the competitive equilibrium strategies derived (as the limit of an n-person Nash equilibrium as $n\to\infty$) in Reinganum (1979).

3.

Suppose there is a pool of oil underlying the property of n people (players). If they sell the oil competitively (or oligopolistically--see example 1) at price p per barrel and if extraction costs depend upon the number of barrels extracted η_i and upon the remaining reserves X(t), then we can formulate the payoffs as

$$V^i(\eta_1,\ldots,\eta_n) = \int_0^T e^{-rt}[p\eta_i - c(t,X,\eta_i)]dt$$

where $\dot{X} = -\Sigma\eta_j$.

Suppose that $\lim_{X\to 0} \frac{\partial c}{\partial \eta_i}(t,X,\eta_i) = \infty$ so that the resource will never be completely exhausted. Then we need not explicitly consider $X(t) \geq 0$.

Again, at a rational threats equilibrium

$$p = \frac{\partial c}{\partial \eta_i}(t,X,\eta_i^*) \qquad i = 1,2,\ldots,n$$

This coincides with myopic, instantaneously competitive behavior.

IV. Conclusions

Although the restrictions in (1) seem fairly strong, several interesting economic problems fit comfortably within the bounds. It is not clear that the restrictions (1) are necessary--they are merely sufficient to generate the result that the threat equilibrium coincides with the (instantaneously) competitive equilibrium.

Recall that these threat strategies are never actually meant to be played--they are merely threats advanced during the bargaining process. But it is interesting to note that, for example, oligopolists will--in an effort to obtain a greater share of the joint profits -- threaten to behave competitively. If bargaining breaks down for some reason, then -- because the threats are assumed to be binding -- the cartel will degenerate to perfectly competitive (not merely Nash-Cournot) behavior until bargaining can be successfully completed.

FOOTNOTES

1. The reader should be cautioned that I am merely seeking the nature of the threats which determine the disagreement point in a game which involves explicit cooperation and bargaining. This ought not to be confused with the analysis of repeated games (or "supergames") as in Friedman (1971), wherein Pareto optima are supported by essentially noncooperative play if strategies of the form "I will play (single period) cooperatively so long as you do; if you ever deviate, I will play (single period) noncooperatively forever" are allowed.

   The games described in this note are not repeated games; they are "one shot" games which may depend on the evolution of an economic variable (or stock) through time. Thus this result has nothing to do with (and claims no applicability to) the repeated games problem.

2. Given s_m, $m \neq i$, the problem is an n-state, 1 control problem for player i. Since H_i is jointly concave in (x, s_i), the first-order conditions are necessary and sufficient to determine a maximizing control (if one exists).

3. For infinite horizon problems, $-\int^t \delta(\tau)d\tau$ may approach $+\infty$ as $t \to \infty$. Thus the analogous transversality condition $\lim_{t\to\infty} \lambda(t) = 0$ provides no information. To deal with this

difficulty, we define the infinite horizon game to be the limit of a sequence of finite horizon games as the horizon grows without bound. Then the solution to the infinite horizon game is defined to be the limit of the finite horizon solution as $T \to \infty$ (if such exists). This procedure is appropriate given certain growth restrictions (see Seierstad, 1977).

REFEREMCES

Friedman, Avner, "Differential Games," New York: John Wiley and Sons, 1971.

Friedman, James, A noncooperative equilibrium for supergames, Review of Economic Studies 38 (1971):1-12

Harsanyi, John C., A bargaining model for the cooperative n-person game. In "Contributions to the Theory of Games IV," edited by A. W. Tucker and R. D. Lucas, Princeton, New Jersey: Princeton University Press, 1959.

Nash, John F., Jr., Two-person cooperative games, Econometrica 21 (1953):128-140.

Reinganum, Jennifer F., Dynamic Games of R & D with rivalry, Ph.D. Dissertation, Northwestern University, 1979.

Seierstad, Atle, Transversality conditions for control problems with infinite horizons, Institute of Economics, University of Oslo, Norway, January 1977.